Exercise 29

Find the limit or show that it does not exist.

$$\lim_{x\to\infty}\left(\sqrt{x^2+ax}-\sqrt{x^2+bx}\right)$$

Solution

Multiply the numerator and denominator by the complex conjugate.

$$\begin{aligned}
\lim_{x\to\infty}\left(\sqrt{x^2+ax}-\sqrt{x^2+bx}\right) &= \lim_{x\to\infty}\left(\sqrt{x^2+ax}-\sqrt{x^2+bx}\right)\cdot\frac{\sqrt{x^2+ax}+\sqrt{x^2+bx}}{\sqrt{x^2+ax}+\sqrt{x^2+bx}} \\
&= \lim_{x\to\infty}\frac{\left(\sqrt{x^2+ax}-\sqrt{x^2+bx}\right)\left(\sqrt{x^2+ax}+\sqrt{x^2+bx}\right)}{\sqrt{x^2+ax}+\sqrt{x^2+bx}} \\
&= \lim_{x\to\infty}\frac{(x^2+ax)-(x^2+bx)}{\sqrt{x^2+ax}+\sqrt{x^2+bx}} \\
&= \lim_{x\to\infty}\frac{ax-bx}{\sqrt{x^2+ax}+\sqrt{x^2+bx}} \\
&= \lim_{x\to\infty}\frac{x(a-b)}{\sqrt{x^2\left(1+\frac{a}{x}\right)}+\sqrt{x^2\left(1+\frac{b}{x}\right)}} \\
&= \lim_{x\to\infty}\frac{x(a-b)}{x\sqrt{1+\dfrac{a}{x}}+x\sqrt{1+\dfrac{b}{x}}} \\
&= \lim_{x\to\infty}\frac{x(a-b)}{x\left(\sqrt{1+\dfrac{a}{x}}+\sqrt{1+\dfrac{b}{x}}\right)} \\
&= \lim_{x\to\infty}\frac{a-b}{\sqrt{1+\dfrac{a}{x}}+\sqrt{1+\dfrac{b}{x}}} \\
&= \frac{\lim\limits_{x\to\infty}(a-b)}{\lim\limits_{x\to\infty}\sqrt{1+\dfrac{a}{x}}+\lim\limits_{x\to\infty}\sqrt{1+\dfrac{b}{x}}} \\
&= \frac{a-b}{\sqrt{1+0}+\sqrt{1+0}} \\
&= \frac{1}{2}(a-b)
\end{aligned}$$